Mobius 函数 & 反演

$\mu: N_+ \longrightarrow \{1, -1, 0\}$
$$\begin{cases} \mu(1) = 1. \\ \mu(n) = \mu(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}) = (-1)^k & r_i = 1 \\ \mu(n) = \mu(p_1^{r_1} \cdots p_k^{r_k}) = 0 & \exists\, r_i \geq 2 \text{ 即有平方} \end{cases}$$

eg: $\mu(3) = -1$. $\mu(15) = \mu(3\times5) = (-1)^2 = 1$ $\mu(6) = 1$. $\mu(12) = \mu(2^2\times3) = 0$

· Mobius 是积性函数，但非完全积性.

$\gcd(m,n) = 1 \Rightarrow \mu(mn) = \mu(m)\mu(n)$ $\mu(p^2) = 0 \neq \mu(p)\cdot\mu(p) = 1$

· 数论函数卷积 $(f*g)(n) = \sum_{d|n} f(d)\, g(\frac{n}{d}) = \sum_{de=n} f(d) g(e) = (g*f)(n)$

$\langle N_+, | \rangle$ 整除偏序集.

** 交换、结合: $f*g*h = \sum_{abc=n} f(a)\, g(b)\, h(c)$

* 单位元 $I(n) = \begin{cases} 1. & n=1. \\ 0 & n\geq 2. \end{cases}$ 则 $\forall f$. $f*I(n) = \sum_{de=n} f(d) I(e) = \sum_{\substack{de=n \\ e=1}} f(d) I(e) = f(n)$

Id

def $e(n) \equiv 1$. $e*e = \sum_{d|n} e(d) e(\frac{n}{d}) = \sum_{d|n} 1 = \tau$ 正因数的个数 $= |\{(a,b)\ ab=n\}|$

$e*e*\overset{k}{e}*\cdots*e$ 将 n 拆成 k 份的方案 ($\langle N_+,|\rangle$ 中 $[1, n]$ 元素个数)

☆: $\sum_{d|n} \mu(d) = \begin{cases} 1. & n=1. \\ 0 & n\geq 2 \end{cases} = I(n)$ 就是单位元!!!

$= e*\mu(n)$

注: (I 是幺元. $e\equiv1$ 是一个特殊常数函数，就是 μ 的逆元).

卷积仍积性 (不一定完全积性)

即: $e*\mu = I(n)$ 得 e 和 μ 在 Dirichlet 卷积下互为逆元.

μ 积性 $\sum_{d|n} \mu(d) = g(n)$ 也是积性. $\begin{cases} g(1) = \mu(1) = 1. & \text{之后. } \mu(a_1\cdots) = 1 \text{ 因仅一项} \\ n\geq 2: g(p_1^{\alpha_1}\cdots p_k^{\alpha_k}) = g(p_1^{\alpha_1})\cdots g(p_k^{\alpha_k}) \end{cases}$

需证 $\sum_{d|p^k} \mu(d) = 0$. $k\geq 1$. $= \mu(1) + \mu(p) + 0 + \cdots + 0 = 1-1 = 0$

则 $n\geq 2$ 时: $g(p_1^{\alpha_1}\cdots p_k^{\alpha_k}) = 0$. 证 $e*\mu = I(n)$

Mobius 反演:

$f(n) = \sum_{d|n} g(d) \iff g(d) = (\mu*f)(n) = \sum_{d|n} \mu(\frac{n}{d}) f(d)$

$\overset{\shortparallel}{e*g(n)}$ $f = e*g \iff g = \mu * f$

$\Rightarrow$. $f = e * g$, $f * \mu = (\mu * 1) * g = I * g = g$!!!

$\Leftarrow$. $g = \mu * f$ $e * \mu = e * \mu * f = I * f = f$

· 算术(数论)函数 $f: N_1 \to \mathbb{C}$

$f(mn) = f(m)f(n)$ 完全乘性. | $f(m+n) = f(m) + f(n)$ 完全加性

$f(1) = 1$. $f(p_1^{\alpha_1} p_2^{\alpha_2} \cdots) = f(p_1)^{\alpha_1} f(p_2)^{\alpha_2} \cdots$ | $\Rightarrow f(1) = 0$.

· 乘性. $(m,n) = 1 \Rightarrow f(mn) = f(m) f(n)$ $f(1) = 1$

$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ $f(n) = f(p_1^{\alpha_1}) \cdots f(p_k^{\alpha_k})$ 不可再拆.

· 完全乘性需 $f(p)$. 乘性需知道 $f(p^k)$

· $I(n)$ 完全乘性. $\mu(n)$ 是乘性函数.

$\varphi(n) = \#\{1 \le k \le n, (k,n) = 1\} = \sum_{\substack{1 \le k \le n \\ (k,n)=1}} 1$ 是乘性函数. $\varphi(n)$ 是乘性,但非完全乘性

$[n]$ 中全体单位 $= \{k: 1 \le k \le n, (k,n) = 1\}$ $\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$

$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ $(\mathbb{Z}/n\mathbb{Z})^* \cong \langle \prod_{i=1}^{k} (\mathbb{Z}/p^{\alpha_i}\mathbb{Z}), \times \rangle \Rightarrow \varphi(n) = \prod \varphi(p_i^{\alpha_i})$

☆ 证 $\varphi * e = Id$ 目标 $(e \equiv 1)$ $(Id(n) = n)$ $= \prod p_i^{\alpha_i - 1}(p_i - 1)$.

即证 $n = \sum_{d|n} \varphi(d) = \sum_{d|n} \varphi(\frac{n}{d})$

MS证 $n = \sum_{1 \le k \le n} 1$ 记 $d = \gcd(n,d)$ $= \sum_{d|n} \sum_{\substack{1 \le k \le n \\ (k,n)=d}} 1 = \sum_{d|n} \varphi(\frac{n}{d}) = \sum_{d|n} \varphi(d) = \varphi * e$

$(\frac{k}{d}, \frac{n}{d}) = 1$ $id(n) = n$

$\boxed{\text{即 } \varphi * e = id}$. $e * \mu = I$. $\Rightarrow \varphi * 1 * \mu = Id * \mu \Rightarrow \varphi = id * \mu$

$\Rightarrow \varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = n \sum_{d|n} \frac{\mu(d)}{d}$

☆ $\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$ $\varphi = id * \mu$ 其中 id 与 μ 均乘性 $\Rightarrow \varphi$ 是乘性.

乘性 $\Rightarrow \varphi(n) = \varphi(p_1^{\alpha_1}) \varphi(p_2^{\alpha_2}) \cdots \varphi(p_k^{\alpha_k})$

之前已证的性质 $= (p_1^{\alpha_1} - p_1^{\alpha_1 - 1}) \cdots (p_k^{\alpha_k} - p_k^{\alpha_k - 1})$

$= n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_k})$ ☆ 即 $\frac{\varphi(n)}{n} = \prod_{i=1}^{k} (1 - \frac{1}{p_k})$

即 $\prod_{p|n} (1 - \frac{1}{p})$

☆: $\frac{\varphi(n)}{n} = \prod_{p|n} (1 - \frac{1}{p})$

$= \sum \frac{\mu(d)}{}$

（$\frac{\varphi(d)}{d}$ 还没乘起的）

$$\sum_{d|n}\frac{\varphi(d)}{d} = \frac{\varphi(n)}{n} = \prod_{p|n}\left(1-\frac{1}{p}\right)$$

若f乘性：$\sum_{d|n} f(d) = (1*f)(n)$

$$= \prod_{i=1}^{k}\left(\sum_{d|p_i^{\alpha_i}} f(d)\right)$$

$$= \prod_{i=1}^{k}\left[f(1)+f(p_i)+f(p_i^2)+\cdots+f(p_i^{\alpha_i})\right]$$

$n=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$

$$\sum_{d|n}\frac{\mu(d)}{d} = \prod_{i=1}^{k}\left(\frac{\overset{1}{\mu(1)}}{1}+\frac{\overset{-1}{\mu(p_i)}}{p_i}+\underbrace{\frac{\mu(p_i^2)}{p_i^2}+\cdots+\frac{\mu(p_i^{\alpha_i})}{p_i^{\alpha_i}}}_{\text{"} 0}\right)$$

$$= \prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right) = \prod_{p|n}\left(1-\frac{1}{p}\right) \ !!!$$

$\mu(p^{\alpha_i})=0,\ \alpha_i\ge 2$

有平方因子直接为0

· 局部有限偏序集上的Mobius反演．（后面用在几个集合的并集上）

const 恒等

$\mathbb{1}*\mu = I(n)$ $\quad g=\mathbb{1}*f \Leftrightarrow f=\mu*g$.

偏序集 partially ordered set $\langle X,\le\rangle$ $\begin{cases} x\le x \\ x\le y \ \&\ y\le z \Rightarrow x\le z \\ x\le y \ \&\ y\le x \Rightarrow x=y \end{cases}$

eg: $(\mathbb{N}^*, |)$ 整除．偏序并不保证任两个元素可比大小的

$[n]=\{1\cdots n\}$ $\quad 2^A=\mathcal{P}(A)=\{B \mid B\subset A\}$ power set 幂集

$\langle 2^{[n]}, \subset\rangle$ 包含于～（偏序非全序，只有是关于子集的元素才有意义）

* 闭区间 $\langle X,\le\rangle$ 就是拿个偏序 poset．$a\le b \in X$ $\quad [a,b]=\{c\in X \mid a\le c\le b\}$.

a,b得先满足个偏序关系才有意义

eg: $\langle \mathbb{N}^*, |\rangle$．$[2,12]=\{2,4,6,12\}$

(2,4,6,12): 闭集 $2 <^{4}_{6}> 12$ 自带

poset $\langle X,\le\rangle$局部有限 $\Longleftrightarrow \forall\, a\le b\in X,\ |[a,b]|<\infty$

$\langle X,\le\rangle$ loc fin poset: $\{f: \{[a,b]: a\le b\in X\}\to \mathbb{C}\}$ Incidence Algebra

$f([a,b])=f(a,b)$ $\quad \begin{cases} (f+g)(a,b)=f(a,b)+g(a,b) \\ (\alpha f)(a,b)=\alpha\cdot f(a,b) \end{cases}$

$a\le b.\ (f*g)(a,b)=\sum_{\substack{a\le c\le b\\(c\in X)}} f(a,c)g(c,b)$ 在$[a,b]$闭集中的$f(a,c)g(c,b)$

$(f*g)*h = f*(g*h)$

$$f*(g*h)\ (a,b) = \sum_{a\le c\le b} f(a,c)(g*h)(a,b) = \sum_{a\le c\le b} f(a,c)\sum_{c\le d\le b} g(c,d)h(d,b)$$

$$= \underbrace{\sum_{a\le c\le d\le b}} f(a,c)g(c,d)h(d,b)$$

$= \sum f(a,c)g(c,d)\sum h(d,b)$ ～

再定义一个"关联函数" $|*|*|$ 是 $[a,b]$ 中有序的点个数

$$\zeta(a,b) \equiv 1 \quad \sum_{a\in A} 1 = |[a,b]| \text{集合大小} = \sum_{a\le c\le b} 1. = |*|.$$

$$e(a,b) \equiv \begin{cases} 1 & a=b \\ 0 & a<b. \end{cases} \Leftrightarrow e(n) = \begin{cases} 1 & n=1 \\ 0 & n>0 \end{cases}$$

$$\begin{cases} (f*e)(a,b) = \sum_{a\le c\le b} f(a,c)e(c,b) \xrightarrow{c=b\text{才非0}} f(a,b)\cdot 1. \\ (e*f)(a,b) \xrightarrow{c=a\text{才非0}} 1\cdot f(a,b) \end{cases}$$

e是单位元

$\alpha(f*g) = (\alpha f)*g = f*(\alpha g)$ → 尝试找关联函数的逆元，用于反演

定义Mobius函数 μ s.t. $\zeta*\mu = \mu*\zeta = e$ | $b>a$.

$$\Rightarrow (1*\mu)(a,b) = \begin{cases} 1. & a=b \\ 0 & a<b \end{cases}$$

再定义. $b=a \Rightarrow 1 = (1*\mu)(a,a) = \mu(a,a)$.

$b>a$: $\underline{0 = (1*\mu)(a,b)} = \sum_{a\le c\le b} 1\cdot\mu(c,b) = \sum_{a\le c\le b}\mu(a,c)$.

$$= \mu(a,b) + \sum_{a<c\le b}\mu(c,b) \qquad \| \; \mu(a,b)+\sum_{a\le c<b}\mu(a,c)$$

归纳地去定义: $\mu(a,b) = -\sum_{a<c\le b}\mu(c,b)$

那么: $\mu(a,b) = -\sum_{a\le c<b}\mu(a,c)$

故 $*$: $1*\mu = \mu*1 = e$ $\quad (\mu_1 = e*\mu_1 = (\mu_2*1)*\mu_1 = \mu_2*e = \mu_2$

⇒ Mobius反演 $g = 1*f \Leftrightarrow f = \mu*g$ (群下逆元)

eg: 容斥原理，数论中的Mobius反演.

应用: $\langle X,\le\rangle$ loc fin poset $\mathcal{A} = \{f: [a,b] : a\le b\} \to \mathbb{C}\}$ $f([a,b]) = f(a,b)$

$f*g(a,b) = \sum_{a\le c\le b} f(a,c)g(c,b)$ $\quad e(a,b) = \begin{cases} 1 & a=b \\ 0. & a<b \end{cases}$

$\mu(a,a) = 1$ $\quad \mu(a,b) = \begin{cases} -\sum_{a<c\le b}\mu(c,b) \\ \text{or } -\sum_{a\le c<b}\mu(a,c) \end{cases}$ $\quad 1*\mu = \mu*1 = e$

$g = 1*f \Leftrightarrow f = \mu*g$

$\mu(a,a)=1$: $\overset{1}{\bullet} \frown \overset{-1}{\bullet} \frown \overset{0}{\bullet} \frown \overset{0}{\bullet} \frown \overset{0}{\bullet} \cdots$ 也就是 $\sum_{a\le c\le b}\mu(a,c)=\begin{cases}1 & a=b\\ 0 & a<b\end{cases}$

1. $\langle N^*, \le\rangle$ $\mu(a,a)=1$. $\quad a, a+1, a+2, \cdots, b$

$\mu(a,a+1)=-1$. $\quad \mu(a,a+n)=0 \quad n\ge 2$

$b=a+1 \Rightarrow \mu(a,b)*1=\mu(a,a)+\mu(a,a+1)=0$

$b\ge a+2$ 前面两项和得 0. 之后每项都得是 0.

$$\mu(a,b)=\begin{cases}1. & a=b\\ -1 & a+1=b\\ 0. & \cancel{\text{else}}\ b-a\ge 2\end{cases}$$

$g=1*f$ 即 $g(a,b)=\sum_{a\le c\le b}f(c,b) \Longleftrightarrow f(a,b)=\mu*g$ 即:

$f=\sum_{a\le c\le b}f(a,c)\mu(c,b)$. $\qquad f=\sum_{a\le c\le b}\mu(a,c)g(c,b)$

只有 $a=c$ $a+1=c$: $f(a,b)=g(a,b)-g(a+1,b)$

$$g(a,b)=\sum_{a\le c\le b}f(c,b) \Longleftrightarrow f(a,b)=g(a,b)-g(a+1,b)$$

eg $g(n)=\sum_{i=1}^{n}f(i) \Rightarrow f(n)=\begin{cases}g(n)-g(n-1) & n\ge 1\\ g(1) & n=1\end{cases}$ 即数列中 S_n 与 a_n 的关系.

eg: $\langle N^*, |\rangle$ $\quad a|b$ 时定义 $\mu(a,b)$. $\quad [a,b]$ 中 c: $a|c$ & $c|b \Rightarrow 1|\frac{c}{a}$ & $\frac{c}{a}|\frac{b}{a}$

☆ $a \diamond b$ $\underline{\text{同构}}$ $1 \diamond \frac{b}{a}$ $\quad$ 从 $a\to b$ 和从 $1\to\frac{b}{a}$ 是同构的

$\mu(a,b)=\mu(1,\frac{b}{a}) \Rightarrow \text{def } \mu(n)=\mu(1,n)$

$n=p$: $\mu(1,p)$ $\quad \underset{1}{1} \longrightarrow \underset{-1}{p}$ $\quad \mu(p)=-1$

$\mu(1,p^k)$ $\quad \underset{1}{1}\to\underset{-1}{p}\to\underset{0}{p^2}\to\cdots\to\underset{0}{p^k}$ $=\begin{cases}1. & p=1\\ -1 & p^k\ k=1\\ 0 & k\ge 2\end{cases}$

$1*\mu=e$. $\mu(1,1)=\mu(1)=1$. $\sum_{a\le c\le b}\mu(c,b)=0$ 即 $\sum_{a|c|b}\mu(c,b)$

$=\sum_{\frac{c}{a}|\frac{b}{a}}\mu(\frac{c}{a},\frac{b}{a})$

$=\sum_{d|n}\mu(d)$ $\quad \begin{cases}d=\frac{c}{a}\\ n=\frac{b}{a}\end{cases}$

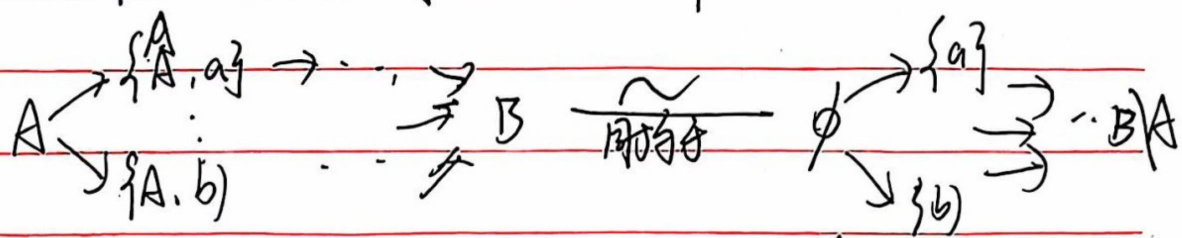
~~Mobius 在数论上~~

$$\mu(n)=\begin{cases}1. & n=1.\\ (-1)^k & n=p_1\cdots p_k\\ 0 & \text{有平方因子.}\end{cases}$$

so that?

·子集格：$[n]=\{1,\cdots n\}$. $2^{[n]}$ 为所有子集. $\langle 2^{[n]},\subset\rangle$

$[A,B]$ 为 all C s.t. $A\subset C\subset B$ 者



$[A,B] \xrightarrow{\text{同构}} [\phi,B\backslash A]$ $\quad A\subset C\subset B \Leftrightarrow \phi\subset C\backslash A \subset B\backslash A$

$\mu(A,B)=\mu(\phi,B\backslash A) \xrightarrow{def} \mu(B\backslash A)$. def: $\mu(A)=\mu(\phi,A)$ 只与$|A|$有关

$|A|=m$. $\mu(\phi,[m])=?$

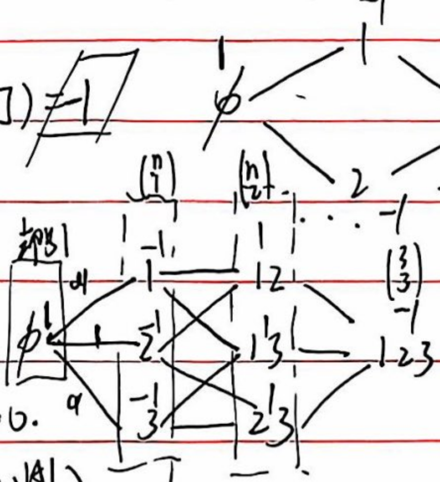
$\mu(\phi)=\mu(\phi,\phi)=\boxed{1}$ $\quad \mu(\phi,[1])=\boxed{-1}$

$\mu(\phi,[2])=1$

$1\cdot\binom{2}{0}+\binom{2}{1}(-1)+\binom{2}{2}1=0$

$\mu(\phi,[3])=-1$

$1\cdot\binom{3}{0}+\binom{3}{1}\cdot -1+\binom{3}{2}1+\binom{3}{3}\cdot -1=0.$



$\Rightarrow \boxed{\mu([n])=(-1)^n}$ $\quad (\mu(A)=(-1)^{|A|})$

·容斥原理：$A_1\cdots A_n$ 集合.

$$|\bigcup A_i|=\sum_{1\le i\le n}A_i-\sum_{i,j}(A_i\cap A_j)+\sum_{i,j,k}\cdots+(-1)^{k-1}\sum_{i_1\cdots i_k}|A_{i_1}\cap\cdots\cap A_{i_k}|$$

★ 用Mobius反演证容斥原理.

$$f(I)=\sum_{J\supseteq I}g(J)\Longleftrightarrow g(I)=\sum_{J\supseteq I}(-1)^{|J|-|I|}f(J)$$

$$f(a)=\sum_{b\ge a}g(b)\Longleftrightarrow g(a)=\sum_{b\ge a}\mu(a,b)f(b)$$

$$\Rightarrow \sum_{b\ge a}\mu(a,b)f(b)=\sum_{b\ge a}\mu(a,b)\sum_{c\ge b}g(c)=\sum_{c\ge a}g(c)\boxed{\sum_{a\le b\le c}\mu(a,b)}$$

$$(1*\mu)(a,c)=\begin{cases}1 & a=c\\ 0 & o/w\end{cases}$$

$$=\sum_{c\ge a}g(c)\,I(a=c)=g(a)$$

示性函数.

集合族中 $\mu(A,B) = (-1)^{|B|-|A|}$

$$f(I) = \sum_{J \supseteq I} g(J) \iff g(I) = \sum_{J \supseteq I} (-1)^{|J|-|I|} f(J)$$

容斥原理：

$$\left|\bigcup A_i\right| = \sum_{\phi \neq I \subseteq [n]} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right| \quad \text{or} = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{I \subseteq [n] \\ |I|=k}} \left|\bigcap_{i \in I} A_i\right|$$

φ有但无影响 φ认为是包括all 的相关的交.

$$\left|\bigcap_{i=1}^{n} A_i^c\right| = \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$$

$$\| \quad |S| - \left|\bigcup_{i=1} A_i\right| \qquad A_\phi = S \qquad A_{I \neq \phi} = \bigcap_{i \in I} A_i$$

$$= |A_\phi| - \sum_{\phi \neq I \subseteq [n]} (-1)^{|I|+1} |A_I|$$

def 中的补集 A_ϕ 为全集

$$= \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$$ 包含了空集

· $f(I) = \sum_{J \supseteq I} g(J) \iff g(I) = \sum_{J \supseteq I} (-1)^{|J|-|I|} f(J)$

def: $f(I) = |A_I| = \begin{cases} |S| & I = \phi \\ \left|\bigcap_{i \in I} A_i\right| & I \neq \phi \end{cases}$

恰属于几个几集的交

在 A_i $i\in I$ 中不在其他中

~~$g(I) = \bigcap_{i \in I} A_i^c$~~

$$B_I = g(I) = \left|\left(\bigcap_{i \in I} A_i\right) \cap \left(\bigcap_{j \notin I} A_j^c\right)\right| \quad \text{eg: } B_{\{1\}} = A_1 \cap A_2^c \cap A_3^c \cap \cdots$$

$B_I \subseteq A_I$ 其中的一部分

两两无交

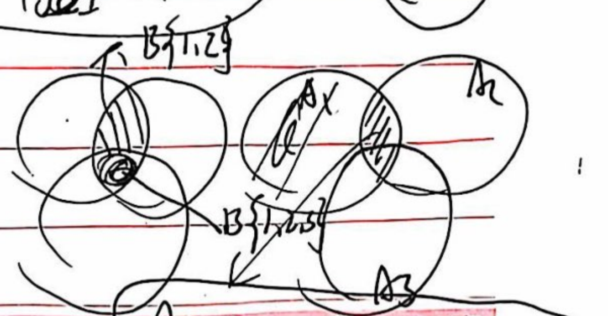
$$\Rightarrow f(I) = \sum_{J \supseteq I} g(J) \quad 即: |A_I| = \sum_{J \supseteq I} |B_J| = \left|\bigcup_{J \supseteq I} B_J\right|$$

$A_I = \bigcup_{J \supseteq I} B_J$ 因为无交并

若 $g(I) = \sum_{J \supseteq I} (-1)^{|J|-|I|} f(J)$

特别的 令 $I = \phi$ $\quad g(\phi) = \left|\bigcap_{i=1}^{n} A_i^c\right|$

$$|B_\phi| = |S| = \sum_{I \subseteq [n]} (-1)^{|I|} \left|\bigcap_{i \in I} A_i\right|$$



$A_{\{1,2\}} = B_{\{1,2\}} \cup B_{\{1,2,3\}}$

容斥原理怎么证:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \cdots + (-1)^{n+1}\left|\bigcap_{i=1}^{n} A_i\right|$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{x\in \bigcup_{i=1}^{n} A_i} 1 = \sum_{\phi\neq I\subseteq[n]} \sum_{x\in B_I} 1$$

LHS: #1

设x在k个中 RHS $= \binom{k}{1} - \binom{k}{2} + \binom{k}{3} \cdots + (-1)^{k+1}\binom{k}{k} + 0 \cdots$

$= 0 + \binom{k}{0} = 1.$ 由(二项式定理)

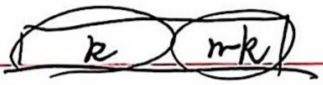
每个元素均成立 $\Rightarrow$ LHS = RHS.

or 直接数. 筛法.

· 组合恒等式 $\binom{n}{k} = |\{A \subset [n] ; |A| = k\}|$

组合证明: 组合意义

$\binom{n}{k} = \binom{n}{n-k}$



$\sum_{k=0}^{n}\binom{n}{k} = 2^n.$

LHS: $[n]$ 子集个数

RHS: 对于每个元素: 选在不在其中

$k\binom{n}{k} = n\binom{n-1}{k-1}$. or $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

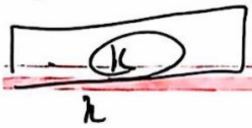
LHS: $\binom{k}{1}\binom{n}{k}$ 选一个k人小队再从中选一队长

RHS: $\binom{n}{1}\binom{n-1}{k-1}$. 选1个队长再从剩 $n-1$ 人中选 $k-1$ 个队友.

Even $= \binom{n}{k-1}\binom{n-k+1}{1}$ 选一个小队再找一个队长.

+: $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$. "选m个队长" $= \binom{n}{k-m}\binom{n-k+m}{m}$.

*: $1\le k<n$: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$



对 i: 选i种: $\binom{n-1}{k-1}$.

没选i种: $\binom{n-1}{k}$.

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-2}{k-1}+\cdots+\binom{k-1}{k-1}$$

→ 最靠前的元素

~~第一个元素是谁？~~ → 考虑将其变得有序，第一个

第一个元素是 1: $\binom{n-1}{k-1}$

2: $\binom{n-2}{k-1}$

⋮

$n-k+1$: $\binom{k-1}{k-1}$

之后无意义

Vandermonde 恒等式：$\binom{m+n}{k}=\sum_{l}\binom{m}{l}\binom{n}{k-l}$ 组合意义

可推广

$$\sum_{k=0}^{n}k\binom{n}{k}=n2^{n-1}$$

（求导可算）

$\binom{k}{1}\binom{n}{k}$

n中选k, 再选一个队长

~~每个队员人数……~~

先选队长 + 任选队员 $\binom{n}{1}2^{n-1}$

⇒ 推广: $\sum_{k=m}^{n}\binom{n}{k}\binom{k}{m}=\binom{n}{m}2^{n-m}$

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k}=(-1)^m\binom{n-1}{m}$$

$\Downarrow$

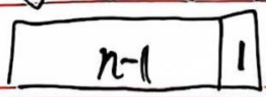
$$\binom{n-1}{0}-\left[\binom{n-1}{0}+\binom{n-1}{1}\right]+\left[\binom{n-1}{1}+\binom{n-1}{2}\right]+\cdots+(-1)^m\left[\binom{n-1}{m-1}+\binom{n-1}{m}\right]$$

$$=(-1)^m\binom{n-1}{m}$$

$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$ ☆ 奇数项的和 = 偶数项的和

n为奇：(…) 选k与n-k一一对应 $\binom{n}{k}=\binom{n}{n-k}$

n为偶：构造双射：



$$|\{S : |S| \text{ odd}, S\subset[n-1]\}|=2^{n-2}$$
$$\|$$
$$|\{S : |S| \text{ even}, S\subset[n-1]\}|=2^{n-2}$$